(#) Riješiti matričnu jednačinu

$$X\cdot(-2)=\begin{bmatrix}1&0\\0&1\end{bmatrix}+X\begin{bmatrix}1&2&3\\-4&-5&-6\end{bmatrix}\begin{bmatrix}-1&-8\\7&1\\-2&-4\end{bmatrix}$$

Rj-upute:

Označimo sa A matricu $A=\begin{bmatrix}1&2&3\\-4&-5&-6\end{bmatrix}\begin{bmatrix}-1&-8\\7&1\\-2&-4\end{bmatrix}=\begin{bmatrix}7&-18\\-19&51\end{bmatrix}$

Imamo

$$X\cdot(-2)=I+XA$$

$$X\cdot(-2)-XA=I$$

$$X(-2I-A)=I \qquad /\cdot(-2I-A)^{-1} \text{ sa desne strane}$$

$$X=(-2I-A)^{-1}$$

Neka je $D=-2I-A=\begin{bmatrix}-2&0\\0&-2\end{bmatrix}-\begin{bmatrix}7&-18\\-19&51\end{bmatrix}=\begin{bmatrix}-9&18\\19&-53\end{bmatrix}$

Kako je

$$D^{-1}=\frac{1}{\det D}D_{kof}^{T}=\frac{1}{\det D}D_{adj}$$

to je $\det(D)=135$

$$D^{-1}=\begin{bmatrix}-\frac{53}{135}&-\frac{2}{15}\\-\frac{19}{135}&-\frac{1}{15}\end{bmatrix}$$

pa je $X=\begin{bmatrix}-\frac{53}{135}&-\frac{2}{15}\\-\frac{19}{135}&-\frac{1}{15}\end{bmatrix}$ traženo rješenje

Riješiti matričnu jednačinu

$$X \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ -7 & -1 \\ 2 & 4 \end{bmatrix} = X \cdot (-1) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rj. - upute

Označimo sa B matricu

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ -7 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -18 \\ -19 & 51 \end{bmatrix}$$

Sad imamo

$$X B = X \cdot (-1) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X B - X \cdot (-1) = I$$

$$X (B + I) = I \quad / \cdot (B+I)^{-1} \text{ sa desne strane}$$

$$X = (B + I)^{-1}$$

Ako sa A označimo matricu $A = B + I$ imamo

$$A = \begin{bmatrix} 8 & -18 \\ -19 & 52 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} A_{kof}^{T} = \frac{1}{\det A} A_{adj}$$

$$\det(A) = 74$$

$$A^{-1} = \begin{bmatrix} \frac{26}{37} & \frac{9}{37} \\ \frac{19}{74} & \frac{4}{37} \end{bmatrix}$$

to je $X = \begin{bmatrix} 26/37 & \frac{9}{37} \\ \frac{19}{74} & \frac{4}{37} \end{bmatrix}$ traženo rješenje

(#) Riješiti matričnu jednačinu

$$\begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 7 & -1 \\ 2 & -4 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2X$$

Rj. - upute

Označimo sa A matricu

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 7 & -1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ -19 & -51 \end{bmatrix}$$

Sad imamo

$$AX = I - 2X$$

$$AX + 2X = I$$

$$(A+2I)X = I \qquad /(A+2I)^{-1} \text{ sa lijeve strane}$$

$$X = (A+2I)^{-1}$$

Ako matricu $A+2I$ označimo sa B imamo $B = \begin{bmatrix} 9 & 18 \\ -19 & -49 \end{bmatrix}$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T} = \frac{1}{\det B} \cdot B_{adj}$$

$$\det(B) = -99$$

$$B^{-1} = \begin{bmatrix} \frac{49}{99} & \frac{2}{11} \\ -\frac{19}{99} & -\frac{1}{11} \end{bmatrix}$$

to je $X = \begin{bmatrix} \frac{49}{99} & \frac{2}{11} \\ -\frac{19}{99} & -\frac{1}{11} \end{bmatrix}$ traženo rješenje

(#) Riješiti matričnu jednačinu

$$3X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 8 \\ -7 & 1 \\ 2 & 4 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rj.-upute

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 8 \\ -7 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ -27 & 61 \end{bmatrix}$$

Ako ovu matricu označimo sa A imamo

$$3X = AX + I$$

$$3X - AX = I$$

$$(3I - A)X = I \quad /(3I-A)^{-1} \text{ sa lijeve strane}$$

$$X = (3I - A)^{-1}$$

$$3I - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -9 & 22 \\ -27 & 61 \end{bmatrix} = \begin{bmatrix} 12 & -22 \\ 27 & -58 \end{bmatrix}$$

$$\det(3I - A) = -102$$

Ako sa C označimo matricu $3I - A$ znamo

$$C^{-1} = \frac{1}{\det C} \cdot C_{kof}^{T} = \frac{1}{\det C} \cdot C_{adj}$$

$$C = \begin{bmatrix} 12 & -22 \\ 27 & -58 \end{bmatrix}$$

$$\det C = -102$$

$$C^{-1} = \begin{bmatrix} \frac{29}{51} & -\frac{11}{51} \\ \frac{9}{34} & -\frac{2}{17} \end{bmatrix}$$

to je $X = \begin{bmatrix} \frac{29}{51} & -\frac{11}{51} \\ \frac{9}{34} & -\frac{2}{17} \end{bmatrix}$ traženo rješenje

Bez upotrebe H'Lopitalovog pravila izračunati sljedeće limese

a) $\lim\limits_{x\to -2} \dfrac{-5x^2-30x-40}{-3x^2+6x+24}$

b) $\lim\limits_{x\to -4} \dfrac{-4x^2-12x+16}{-2x^2-10x-8}$

c) $\lim\limits_{x\to -6} \dfrac{3x^2+12x-36}{2x^2+10x-12}$

d) $\lim\limits_{x\to -8} \dfrac{5x^2+35x-40}{-2x^2-6x+80}$

Rj.

a) $$\lim_{x\to -2} \frac{-5x^2-30x-40}{-3x^2+6x+24} = \lim_{x\to -2} \frac{(-5)\cancel{(x+2)}(x+4)}{(-3)\cancel{(x+2)}(x-4)} = \frac{-5\cdot 2}{-3\cdot(-6)} = -\frac{10}{18} = -\frac{5}{9}$$

b) $$\lim_{x\to -4} \frac{-4x^2-12x+16}{-2x^2-10x-8} = \lim_{x\to -4} \frac{(-4)\cancel{(x+4)}(x-1)}{(-2)\cancel{(x+4)}(x+1)} = \frac{(-4)\cdot(-5)}{(-2)(-3)} = \frac{20}{6} = \frac{10}{3}$$

c) $$\lim_{x\to -6} \frac{3x^2+12x-36}{2x^2+10x-12} = \lim_{x\to -6} \frac{3\cancel{(x+6)}(x-2)}{2\cancel{(x+6)}(x-1)} = \frac{3\cdot \overset{-4}{\cancel{(-8)}}}{\cancel{2}\cdot(-7)} = \frac{12}{7}$$

d) $$\lim_{x\to -8} \frac{5x^2+35x-40}{-2x^2-6x+80} = \lim_{x\to -8} \frac{5\cancel{(x+8)}(x-1)}{(-2)\cancel{(x+8)}(x-5)} = \frac{5\cdot(-9)}{(-2)(-13)} = -\frac{45}{26}$$

(#) Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x}{x^2-1}$$

Rj.-upute

DEFINICIONO PODRUČJE

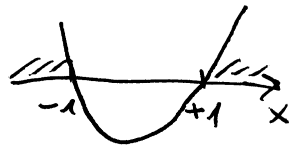
$x^2-1 \neq 0 \quad \wedge \quad \frac{x}{x^2-1} > 0$

$x^2 \neq 1$

$x \neq \pm 1$

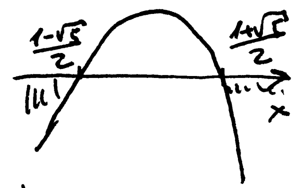
$\mathcal{D}: x \in (-1,0) \cup (1,+\infty)$

$x^2-1=0$



x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,+\infty)$
x	$-$	$-$	$+$	$+$
x^2-1	$+$	$-$	$-$	$+$
$\frac{x}{x^2-1}$	$-$	$+$	$-$	$+$

$-x^2+x+1$



ZNAK

$\ln \frac{x}{x^2-1} > 0$

$\ln \frac{x}{x^2-1} > \ln 1$

$\frac{x}{x^2-1} > 1$

$\frac{x}{x^2-1} - 1 > 0$

$\frac{x-(x^2-1)}{x^2-1} > 0$

$\frac{-x^2+x+1}{x^2-1} > 0$

$\frac{(-1)(x^2-x-1)}{x^2-1} > 0$

$x^2-x-1 > 0 \qquad D = 1+4 = 5$

$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

x	$(-1, \frac{1-\sqrt{5}}{2})$	$(\frac{1-\sqrt{5}}{2}, 0)$	$(1, \frac{1+\sqrt{5}}{2})$	$(\frac{1+\sqrt{5}}{2}, +\infty)$
Y	$+$	$-$	$+$	$-$

znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x}{x^2-1}\right)' = \frac{x^2+1}{-x^3+x} = \frac{x^2+1}{(-x)(x^2-1)}$$

x	$(-1,0)$	$(1,+\infty)$
Y'	$-$	$-$
Y	$\searrow$	$\searrow$

F-ja uvijek opada pa nema ekstrema.

Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x-1}{x^2+1}$$

Rj. - upute

DEFINICIONO PODRUČJE

x^2+1 je pozitivno za svako $x \in \mathbb{R}$

pa je $\frac{x-1}{x^2+1} > 0$ akko $x-1>0$

tj. za $x>1$

$\mathcal{D}: \ x \in (1, +\infty)$

$x > 1$

ZNAK

$$\ln \frac{x-1}{x^2+1} > 0$$

$$\ln \frac{x-1}{x^2+1} > \ln 1$$

$$\frac{x-1}{x^2+1} > 1$$

$$\frac{x-1}{x^2+1} - 1 > 0$$

$$\frac{x-1-(x^2+1)}{x^2+1} > 0$$

$$\frac{-x^2+x-2}{x^2+1} > 0$$

$$\frac{(-1)(x^2-x+2)}{x^2+1} > 0$$

Kako je $x^2-x+2>0 \ \forall x$

to je $(-1)(x^2-x+2)<0 \ \forall x$

x	$(1,+\infty)$
y	$-$

znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x-1}{x^2+1}\right)' = - \frac{x^2-2x-1}{(x^2+1)(x-1)}$$

$$x^2-2x-1=0$$

$$D = 4+4 = 8$$

$$x_{1,2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$$x_1 = 1-\sqrt{2} \notin \mathcal{D} \qquad x_2 = 1+\sqrt{2} \in \mathcal{D}$$

x	$(1, 1+\sqrt{2})$	$(1+\sqrt{2}, +\infty)$
y'	$+$	$-$
y	↗	↘

MAX

tabela rasta i opadanja

$$f(1+\sqrt{2}) = \ln \frac{1+\sqrt{2}-1}{(1+\sqrt{2})^2+1} = \ln \frac{\sqrt{2}}{1+2\sqrt{2}+2+1} = \ln \frac{\sqrt{2}}{4+2\sqrt{2}}$$

F-ja ima ekstrem i to maksimum u tački $\left(1+\sqrt{2}; \ \ln \frac{\sqrt{2}}{4+2\sqrt{2}}\right)$

Ⓐ Odrediti definiciono područje, znak te ekstreme f-je

$$y=\ln\frac{x^2-1}{x+1}.$$

Rj.-upute

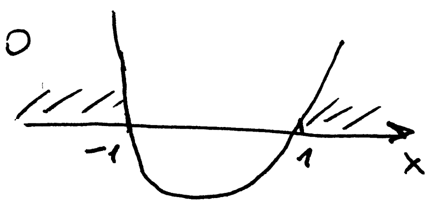
DEFINICIONO PODRUČJE

$x+1\neq 0 \quad \wedge \quad \frac{x^2-1}{x+1}>0$

$x\neq -1$

$$\frac{(x-1)(x+1)}{x+1}>0$$

$x^2-1>0$



x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
x^2-1	$+$	$-$	$+$
$x+1$	$-$	$+$	$+$
$\frac{x^2-1}{x+1}$	$-$	$-$	$+$

$\mathcal{D}:\ x\in(1,+\infty)$

$x>1$

ZNAK

$$\ln\frac{x^2-1}{x+1}>0$$

$$\ln\frac{x^2-1}{x+1}>\ln 1$$

$$\frac{x^2-1}{x+1}>1$$

$$\frac{x^2-1}{x+1}-1>0$$

$$\frac{x^2-1-x-1}{x+1}>0$$

$$\frac{x^2-x-2}{x+1}>0$$

$$\frac{(x-2)(x+1)}{x+1}>0$$

$2,-1$

x	$(2,+\infty)$
$\frac{x^2-x-2}{x+1}$	$+$

x	$(1,2)$	$(2,+\infty)$
y	$-$	$+$

Znak f-je

EKSTREMI F-JE

$$y'=\left(\ln\frac{x^2-1}{x+1}\right)'=\frac{1}{x-1}$$

x	$(1,+\infty)$
y'	$+$
y	$\nearrow$

tabela rasta i opadanja

F-ja uvijek raste pa nema ekstrema.

(#) Odrediti definiciono područje, znak te ekstreme f-je

$$y = \ln \frac{x+1}{x-1}.$$

Rj.-upute:

DEFINICIONO PODRUČJE

$x-1 \neq 0$ $\qquad$ $\frac{x+1}{x-1} > 0$

$x \neq 1$

$\mathcal{D}: x \in (-\infty, -1) \cup (1, +\infty)$

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
$x+1$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
$\frac{x+1}{x-1}$	$+$	$-$	$+$

ZNAK

$\ln \frac{x+1}{x-1} > 0$ $\qquad$ $\frac{x+1}{x-1} - 1 > 0$ $\qquad$ $x-1 > 0$

$\qquad$ $x > 1$

$\ln \frac{x+1}{x-1} > \ln 1$ $\qquad$ $\frac{x+1-x+1}{x-1} > 0$

$\frac{x+1}{x-1} > 1$ $\qquad$ $\frac{2}{x-1} > 0$

x	$(-\infty,-1)$	$(1,+\infty)$
y	$-$	$+$

znak f-je

EKSTREMI F-JE

$$y' = \left(\ln \frac{x+1}{x-1}\right)' = -\frac{2}{x^2-1}$$

x	$(-\infty,-1)$	$(1,+\infty)$
y'	$-$	$-$
y	$\searrow$	$\searrow$

tabela rasta i opadanja

F-ja uvijek opada pa nema ekstrema.

(#) Izračunati sljedeće integrale

a) $\int x \ln(x-1)\,dx$ c) $\int \ln(x^2-1)\,dx$

b) $\int \ln(1+x^2)\,dx$ d) $\int (x+1)\ln x\,dx$

Rj.

a) $$\int x\ln(x-1)\,dx = \left| \begin{matrix} u=\ln(x-1) & dv = x\,dx \\ du=\frac{1}{x-1}dx & v=\frac{1}{2}x^2 \end{matrix} \right| = \frac{1}{2}x^2\ln(x-1) -$$

$$-\frac{1}{2}\int \frac{x^2}{x-1}\,dx = \left| \frac{x^2}{x-1} = \frac{x^2-1+1}{x-1} = \frac{(x-1)(x+1)+1}{x-1} = x+1+\frac{1}{x-1} \right|$$

$$= \frac{1}{2}x^2\ln(x-1) - \frac{1}{2}\left(\frac{1}{2}x^2 + x + \ln|x-1|\right) + C = \frac{x^2-1}{2}\ln|x-1| - \frac{x^2}{4} - \frac{x}{2} + C$$

b) $$\int \ln(1+x^2)\,dx = \left| \begin{matrix} u=\ln(1+x^2) & dv=dx \\ du=\frac{1}{1+x^2}\cdot 2x\,dx & v=x \end{matrix} \right| =$$

$$= x\ln(1+x^2) - 2\int \frac{x^2 \overset{+1-1}{}}{1+x^2}\,dx =$$

$$= x\ln(1+x^2) - 2\int\left(1-\frac{1}{1+x^2}\right)dx =$$

$$= x\ln(1+x^2) - 2x + 2\,\mathrm{arctg}\,x + C$$

c) $$\int \ln(x^2-1)\,dx = \begin{vmatrix} u=\ln(x^2-1) & dv=dx \\ du=\frac{2x}{x^2-1} & v=x \end{vmatrix} = x\ln(x^2-1)-2\int\frac{x^2-1+1}{x^2-1}\,dx$$

$$= x\ln(x^2-1)-2\cdot\int\left(1+\frac{1}{x^2-1}\right)dx =$$

$$= x\ln(x^2-1)-2x-2\cdot\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

$$= x\ln(x^2-1)-2x-\ln\left|\frac{x-1}{x+1}\right|+C$$

d) $$\int (x+1)\ln x\,dx = \begin{vmatrix} u=\ln x & dv=x+1 \\ du=\frac{dx}{x} & v=\frac{1}{2}x^2+x \end{vmatrix} =$$

$$= \left(\frac{1}{2}x^2+x\right)\ln x-\int\left(\frac{1}{2}x+1\right)dx$$

$$= \left(\frac{1}{2}x^2+x\right)\ln x-\frac{1}{4}x^2-x+C$$

\# Odrediti stacionarne tačke f-je

$$z = \frac{1}{2}x^2 - xy + xy^2 - \frac{1}{2}x^2 y$$

Rj.

$$\frac{\partial z}{\partial x} = x - y + y^2 - xy$$

$$\frac{\partial z}{\partial y} = -x + 2xy - \frac{1}{2}x^2$$

$$x - y + y^2 - xy = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

$$x - y + y(y - x) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

$$(x-y) - y(x-y) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0$$

$$(x-y)(1-y) = 0$$
$$-x + 2xy - \frac{1}{2}x^2 = 0 \quad \ldots (*)$$

$x - y = 0$ ili $1 - y = 0$

a) $1 - y = 0$

$y = 1$

$$(*) \Rightarrow \quad -x + 2x - \frac{1}{2}x^2 = 0$$

$$x - \frac{1}{2}x^2 = 0$$

$$x\left(1 - \frac{1}{2}x\right) = 0$$

$x = 0$ ili $x = 2$ $\qquad M_1(0,1),\ M_2(2,1)$

b)

$x - y = 0$

$$x = y \overset{(*)}{\Rightarrow} \quad -x + 2x^2 - \frac{1}{2}x^2 = 0$$

$$-x + \frac{3}{2}x^2 = 0$$

$$x\left(-1 + \frac{3}{2}x\right) = 0$$

$x = 0 \Rightarrow y = 0$

$x = \frac{2}{3} \Rightarrow y = \frac{2}{3}$

$M_3(0;0),\ M_4\left(\frac{2}{3};\frac{2}{3}\right)$

Stacionarne tačke su

$$M_1(0;1),\ M_2(2;1),\ M_3(0;0),\ M_4\left(\frac{2}{3};\frac{2}{3}\right).$$

(#) Odrediti stacionarne tačke f-je

$$z = 9x^2 - \frac{9}{2}x^2y + 6xy^2 - 12xy$$

Rj.

$$\frac{\partial z}{\partial x} = 18x - 9xy + 6y^2 - 12y$$

$$\frac{\partial z}{\partial y} = -\frac{9}{2}x^2 + 12xy - 12x$$

$$18x - 9xy + 6y^2 - 12y = 0 \quad /:3$$
$$-\frac{9}{2}x^2 + 12xy - 12x = 0 \quad /:3$$

$$6x - 3xy + 2y^2 - 4y = 0$$
$$-\frac{3}{2}x^2 + 4xy - 4x = 0 \quad /\cdot 2$$

$$y(-3x + 2y) - 2(2y - 3x) = 0$$
$$-3x^2 + 8xy - 8x = 0$$

$$(y-2)(2y-3x) = 0$$
$$-3x^2 + 8xy - 8x = 0$$

$$y - 2 = 0 \quad \text{ili} \quad 2y - 3x = 0$$

a) $y - 2 = 0$

$y = 2$

$-3x^2 + 8x \cdot 2 - 8x = 0$

$8x - 3x^2 = 0 \quad /:3$

$x(8 - 3x) = 0$

$x = 0$ ili $x = \frac{8}{3}$

$M_1(0; 2)$, $M_2(\frac{8}{3}; 2)$

b) $2y - 3x = 0$

$y = \frac{3}{2}x$

$-3x^2 + \overset{4}{\cancel{8}} x \cdot \frac{3}{\cancel{2}_1} x - 8x = 0$

$-3x^2 + 12x^2 - 8x = 0$

$9x^2 - 8x = 0$

$x(9x - 8) = 0$

$x_1 = 0 \Rightarrow y_1 = 0$

$x_2 = \frac{8}{9} \Rightarrow$

$y_2 = \frac{3}{2} \cdot \frac{8}{9} = \frac{4}{3}$

$M_3(0; 0)$ $M_4(\frac{8}{9}; \frac{4}{3})$

Stacionarne tačke su $M_1(0; 2)$, $M_2(\frac{8}{3}; 2)$, $M_3(0; 0)$ i $M_4(\frac{8}{9}; \frac{4}{3})$

(#) Odrediti stacionarne tačke f-je

$$z = x^2y - \frac{1}{2}xy^2 - xy + \frac{1}{2}y^2$$

Rj.

$$\frac{\partial z}{\partial x} = 2xy - \frac{1}{2}y^2 - y$$

$$\frac{\partial z}{\partial y} = x^2 - xy - x + y$$

$$2xy - \frac{1}{2}y^2 - y = 0$$
$$x^2 - xy - x + y = 0$$

$$2xy - \frac{1}{2}y^2 - y = 0$$
$$x(x-y) - 1\cdot(x-y) = 0$$

$$2xy - \frac{1}{2}y^2 - y = 0 \quad \dots (1)$$
$$(x-y)(x-1) = 0$$

$x - y = 0$ ili $x - 1 = 0$

a) $x - 1 = 0$

$x = 1$

$$(1) \Rightarrow \quad 2y - \frac{1}{2}y^2 - y = 0$$

$$y - \frac{1}{2}y^2 = 0$$

$$y\left(1 - \frac{1}{2}y\right) = 0$$

$y = 0$ ili $y = 2$

$M_1(1;0)$, $M_2(1;2)$

b) $x - y = 0$

$$x = y \overset{(1)}{\Rightarrow} \quad 2x^2 - \frac{1}{2}x^2 - x = 0$$

$$\frac{3}{2}x^2 - x = 0$$

$$x\left(\frac{3}{2}x - 1\right) = 0$$

$x = 0 \Rightarrow y = 0$

$x = \frac{2}{3} \Rightarrow y = \frac{2}{3}$

$M_3(0;0)$, $M_4\left(\frac{2}{3};\frac{2}{3}\right)$

Stacionarne tačke su $M_1(1;0)$, $M_2(1;2)$, $M_3(0;0)$ i $M_4\left(\frac{2}{3};\frac{2}{3}\right)$.

Odrediti stacionarne tačke f-je

$$z = 6x^2y - \frac{9}{2}xy^2 - 12xy + 9y^2$$

Rj.

$$\frac{\partial z}{\partial x} = 12xy - \frac{9}{2}y^2 - 12y$$

$$\frac{\partial z}{\partial y} = 6x^2 - 9xy - 12x + 18y$$

$$12xy - \frac{9}{2}y^2 - 12y = 0 \quad /:3$$
$$6x^2 - 9xy - 12x + 18y = 0 \quad /:3$$

$$4xy - \frac{3}{2}y^2 - 4y = 0 \quad /\cdot 2$$
$$2x^2 - 3xy - 4x + 6y = 0$$

$$8xy - 3y^2 - 8y = 0$$
$$x(2x - 3y) - 2(2x - 3y) = 0$$

$$8xy - 3y^2 - 8y = 0$$
$$(x-2)(2x-3y) = 0$$

$$x - 2 = 0 \quad \text{ili} \quad 2x - 3y = 0$$

a)
$$x - 2 = 0$$
$$x = 2$$
$$8 \cdot 2 \cdot y - 3y^2 - 8y = 0$$
$$8y - 3y^2 = 0$$
$$y(8 - 3y) = 0$$
$$y = 0 \quad \text{ili} \quad y = \frac{8}{3}$$

$$M_1(2; 0), \quad M_2\left(2; \frac{8}{3}\right)$$

b)
$$2x - 3y = 0$$
$$x = \frac{3}{2}y \qquad \cancel{8}^{4} \cdot \frac{3}{\cancel{2}_1}y \cdot y - 3y^2 - 8y = 0$$
$$-8y + 9y^2 = 0$$
$$y(9y - 8) = 0$$

$$y_1 = 0 \Rightarrow x_1 = 0$$
$$y_2 = \frac{8}{9} \Rightarrow x_2 = \frac{3}{2} \cdot \frac{8}{9} = \frac{4}{3}$$

$$M_3(0; 0), \quad M_4\left(\frac{4}{3}; \frac{8}{9}\right)$$

Stacionarne tačke f-je su

$$M_1(2; 0), \quad M_2\left(2; \frac{8}{3}\right), \quad M_3(0; 0), \quad M_4\left(\frac{4}{3}; \frac{8}{9}\right)$$

(#) Riješiti diferencijalnu jednačinu $y - xy' - \frac{1}{2}y'^2 = 0$.

Rj.

$$y - xy' - \frac{1}{2}y'^2 = 0$$

$$y = xy' + \frac{1}{2}y'^2$$

Jednačine oblika $y = xy' + f(y')$ se nazivaju Clairaut-ove dif. jed.

uvodimo smjenu $y' = p$

$$dy = p\,dx$$

$$y = xp + \frac{1}{2}p^2 \quad /d$$

$$\underbrace{dy}_{=p\,dx} = \underbrace{p\,dx}\, + x\,dp + p\,dp$$

$$(x+p)\,dp = 0$$

(a) $dp = 0$

$$p = C$$

$$y = xC + \frac{1}{2}C^2$$

je opšte rješenje diferencijalne jednačine

(b) $x + p = 0$

$$p = -x$$

$$y = xp + \frac{1}{2}p^2$$

$$y = -x^2 + \frac{1}{2}x^2$$

$$y = -\frac{1}{2}x^2$$

singularno rješenje diferencijalne jednačine

\# Riješiti diferencijalnu jednačinu $y'^2 - xy' + y = 0$.

Rj:

$$y'^2 - xy' + y = 0$$

$$y = xy' - y'^2$$

Jednačina oblika $y = xy' + f(y')$ se naziva Clairautova dif. jedn.

uvodimo smjenu $y' = p \Rightarrow dy = p\,dx$

$$y' = \frac{dy}{dx}$$

$$y = xp - p^2 \quad /d$$

$$dy = p\,dx + x\,dp - 2p\,dp$$

$$\cancel{p\,dx} = \cancel{p\,dx} + x\,dp - 2p\,dp$$

$$(x - 2p)\,dp = 0$$

(a) $dp = 0$

$p = c \Rightarrow y = xc - c^2$

je opšte rješenje diferencijalne jednačine

(b) $x - 2p = 0$

$$2p = x$$

$$p = \frac{x}{2}$$

$$y = xp - p^2$$

$$y = \frac{1}{2}x^2 - \frac{1}{4}x^2$$

$$y = \frac{1}{4}x^2$$

singularno rješenje diferencij. jedn.

Riješiti diferencijalnu jednačinu $(y-y'x)^2 = 1+y'^2$.

Rj: $y - y'x = \pm\sqrt{1+y'^2}$

$y = y'x \pm \sqrt{1+y'^2}$

Jednačina oblika $y = xy' + f(y')$ se naziva Clairat-ova dif. jedn. i ove diferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove difer. jednač. uvodimo smjenu $y' = p$. ($dy = p\,dx$, $x = uv$).

$y = y'x \pm \sqrt{1+y'^2}$ ovo je Clairaut-ova dif. jedn.

$y' = p$, $y' = \frac{dy}{dx} \Rightarrow dy = p\,dx$

$y = px \pm \sqrt{1+p^2} \quad /d$

$\underbrace{dy}_{=p\,dx} = p\,dx + x\,dp \pm \frac{2p}{2\sqrt{1+p^2}}\,dp$

$\left(x \pm \frac{p}{\sqrt{1+p^2}}\right)dp = 0$

(a) $dp = 0$

$p = c$

$(y - cx)^2 = 1 + c^2$

opšte rješenje diferencijalne jedn.

(b) $x \pm \frac{p}{\sqrt{1+p^2}} = 0$

$\pm \frac{p}{\sqrt{1+p^2}} = -x \quad /^2$

$\frac{p^2}{1+p^2} = x^2$

$p^2 = \underbrace{x^2(1+p^2)}_{x^2 + x^2p^2}$

$(1-x^2)p^2 = x^2$

$p^2 = \frac{x^2}{1-x^2}$

$p = \frac{x}{\sqrt{1-x^2}}$

$y = \frac{x^2}{\sqrt{1-x^2}} \pm \sqrt{1 + \frac{x^2}{1-x^2}} = \frac{x^2 \pm 1}{\sqrt{1-x^2}}$ singularno rješenje diferenc. jednačine

(#) Riješiti diferencijalnu jednačinu $y = y'x + \sqrt{4 + y'^2}$.

Rj. Jednačina oblika $y = xy' + f(y')$ se naziva Clairaut-ova diferencijalna jednačina i ove diferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove diferencijalne jednačine – uvodimo smjenu

$$y' = p, \quad \left[y' = \frac{dy}{dx} \right]$$

$$dy = p\,dx$$

$$x = uv$$

$$y = y'x + \sqrt{4 + y'^2} \quad \text{Clair. dif. jedn.}$$

$$y' = p \Rightarrow dy = p\,dx$$

$$y = px + \sqrt{4 + p^2} \quad /d$$

$$dy = p\,dx + x\,dp + \frac{2p\,dp}{2\sqrt{4 + p^2}}$$

$$\cancel{p\,dx} = \cancel{p\,dx} + x\,dp + \frac{p\,dp}{\sqrt{4 + p^2}}$$

$$\left(x + \frac{p}{\sqrt{4 + p^2}} \right) dp = 0$$

(a) $dp = 0 \Rightarrow p = C$

$$y = Cx + \sqrt{4 + C^2}$$

opšte rješenje diferenc. jednačine

(b) $x + \frac{p}{\sqrt{4 + p^2}} = 0$

$$\frac{p}{\sqrt{4 + p^2}} = -x \quad /^2$$

$$\frac{p^2}{4 + p^2} = x^2$$

$$p^2 = x^2 (4 + p^2)$$

$$p^2 = x^2 p^2 + 4x^2$$

$$(1 - x^2) p^2 = 4x^2$$

$$p^2 = \frac{4x^2}{1 - x^2}$$

$$p = \frac{2x}{\sqrt{1 - x^2}}$$

$$y = px + \sqrt{4 + p^2} = \frac{2x^2}{\sqrt{1 - x^2}} + \sqrt{4 + \frac{4x^2}{1 - x^2}} = \frac{2x^2}{\sqrt{1 - x^2}} + \sqrt{\frac{4 - 4x^2 + 4x^2}{1 - x^2}}$$

$$y = \frac{2x^2 + 2}{\sqrt{1 - x^2}}$$

singularno rješenje diferencijalne jednačine (rješenje koje se ne može dobiti iz opšteg rješenja)